

STRONG APPROXIMATION FOR CERTAIN QUADRIC FIBRATIONS WITH COMPACT FIBERS

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1. INTRODUCTION

In [27], G.L.Watson investigated the local-global principle over \mathbb{Z} for the following equation

$$q(x_1, \dots, x_n) + \sum_{i=1}^n a_i(t)x_i + b(t) = 0 \quad (1.1)$$

where $q(x_1, \dots, x_n)$ is a quadratic form over \mathbb{Z} and $a_1(t), \dots, a_n(t)$ and $b(t)$ are polynomials over \mathbb{Z} . The variety defined by (1.1) is isomorphic to the variety defined by

$$q(x_1, \dots, x_n) = p(t) \quad (1.2)$$

over \mathbb{Q} , where $p(t)$ is a polynomial over \mathbb{Q} . The study of strong approximation of equation (1.2) will provide the solvability of equation (1.1) by choosing the equation (1.1) as an integral model. In [5], J.-L. Colliot-Thélène and the author proved that the equation (1.2) satisfies the strong approximation with Brauer-Manin obstruction when the quadratic form q is indefinite with $n \geq 3$. In [3], J.-L. Colliot-Thélène and D.Harari extended this result to very general fibration of homogeneous spaces of linear algebraic groups with isotropic assumption for the fibers over \mathbb{R} -points. By such fibration method, one needs enough fibers satisfying the strong approximation with Brauer-Manin obstruction. One of the necessary conditions of strong approximation with Brauer-Manin obstruction for subvarieties of affine varieties is that the set of \mathbb{R} -points is not compact. In this paper, we will show that strong approximation with Brauer-Manin obstruction holds for (1.2) with $n \geq 3$ if and only if \mathbb{R} -points of total spaces is not compact. When the quadratic form q is definite, this provides the example that none of fibers satisfies strong approximation with Brauer-Manin obstruction. As application, the main result in [27] is the immediate consequence of our result. Moreover, in order to prove our result, we develop the representation theory of quadratic Diophantine equations and explain that the representability of quadratic polynomials is equivalent to the classical result of representability of quadratic forms with congruent conditions and extend Theorem 2.1 in [7] to general number fields.

Notation and terminology are standard. Let F be a number field, \mathfrak{o}_F be the ring of integers of F and Ω_F be the set of all primes in F . For each $v \in \Omega_F$, let F_v be the completion of F at v . Let ∞_F be the set of archimedean primes in F and write $v < \infty_F$ for $v \in \Omega_F \setminus \infty_F$. For each $v < \infty_F$, let \mathfrak{o}_v be the completion of \mathfrak{o}_F at v and π_v is the uniformizer of \mathfrak{o}_v with $\text{ord}_v(\pi_v) = 1$ and assume $\text{ord}_v(0) = +\infty$. Write $\mathfrak{o}_v = F_v$ for $v \in \infty_F$. For any finite subset T of Ω_F , let $F_T = \prod_{v \in T} F_v$. For any finite subset S of Ω_F containing ∞_F , the S -integers are defined to be

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elements in F which are integral outside S and denoted by \mathfrak{o}_S . Let \mathbb{A}_F be the adelic group of F with its usual topology. For any finite set T of Ω_F , one defines $\mathbb{A}_F^T \subset (\prod_{v \notin T} F_v)$ equipped with analogous adelic topology. The natural projection which omits the T -coordinates defines homomorphism of rings $\mathbb{A}_F \rightarrow \mathbb{A}_F^T$. For any scheme of finite type X over F this induces a map

$$pr^T : X(\mathbb{A}_F) \rightarrow X(\mathbb{A}_F^T).$$

Let $Br(X) = H_{\text{et}}^2(X, \mathbb{G}_m)$ and

$$X_F(\mathbb{A}_F)^{Br(X)} = \{(x_v)_v \in X_F(\mathbb{A}_F) : \sum_{v \in \Omega_F} \text{inv}_v(\xi(x_v)) = 0 \text{ for all } \xi \in Br(X)\}.$$

Then

$$X(F) \subseteq X_F(\mathbb{A}_F)^{Br(X)} \subseteq X_F(\mathbb{A}_F)$$

by class field theory.

Definition 1.3. *Let X be a scheme of finite type over F . One says that strong approximation with Brauer-Manin obstruction off T holds for X if $X(F)$ is dense in $pr^T(X(\mathbb{A}_F)^{Br(X)})$ under the diagonal map.*

The paper is organized as follows. In §2, we set up spinor genus theory for quadratic Diophantine equations briefly. In §3, we proved strong approximation with Brauer-Manin obstruction off ∞_F for the variety defined by (1.2). As application, we explain the equivalence between representability of quadratic Diophantine polynomials and representability of quadratic forms with congruent condition for definite case and extend Theorem 2.1 in [7] over a number field.

2. QUADRATIC DIOPHANTINE EQUATIONS

Arithmetic theory of quadratic forms is a classical topic and has been extensively studied by various methods, for example in [6], [16], [1] and [12]. In order to establish the strong approximation for (1.2) with definite quadratic form q , we need to extend the classical results for quadratic forms to general quadratic polynomials. Such generalization has already been considered by Kneser in [14] and Watson in [26]. Recently, for various purpose, such generalization has been studied by Shimura in [21], [22] and [23], Sun in [25], Colliot-Thélène and Xu in [4], Wei and Xu in [28] and Chan and Oh in [2] and so on. In this section, we briefly set up some basic properties for quadratic Diophantine equations which we need in next sections.

Let V be a non-degenerated quadratic space over F with the associated symmetric bilinear map

$$\langle, \rangle : V \times V \longrightarrow F \quad \text{with} \quad q(x) = \langle x, x \rangle$$

for any $x \in V$ and the special orthogonal group

$$SO(V) = \{\sigma \in GL(V) : q(\sigma x) = q(x) \text{ for all } x \in V \text{ and } \det(\sigma) = 1\}$$

with the double cover $Spin(V)$ satisfying

$$1 \rightarrow \mu_2 \rightarrow Spin(V) \xrightarrow{\iota} SO(V) \xrightarrow{\theta} H^1(F, \mu_2) \cong F^\times / (F^\times)^2 \quad (2.1)$$

by Galois cohomology. Let $SO_{\mathbb{A}}(V)$, $Spin_{\mathbb{A}}(V)$ and $GL_{\mathbb{A}}(V)$ be the adelic groups of $SO(V)$, $Spin(V)$ and $GL(V)$ respectively.

A finitely generated \mathfrak{o}_F -module L in V is called an \mathfrak{o}_F -lattice if $FL = V$. We denote the local completion of a lattice L inside the local completion V_v of V at v by L_v for $v < \infty_F$ and $L_v = V_v$ for $v \in \infty_F$. The classical theory of integral quadratic forms can be formulated in terms of lattices such as [16] and [12]. Most of theory can be extended to general quadratic equations in terms of lattice translations, i.e. $L + u_0$ for some $u_0 \in V$.

The local representation of integral quadratic forms has been extensively studied in [17] and [19]. For general quadratic equations, one has the following result.

Lemma 2.2. *Let L be a lattice in V and $u_0 \in V$. If $u_0 \notin L_v$ for $v < \infty_F$, then $q(L_v + u_0)$ is an open compact subset of F_v .*

Proof. Since the quadratic form q on V_v defines a morphism

$$q : \mathbf{A}^n \rightarrow \mathbf{A}^1$$

of affine spaces over F_v , one obtains that q is smooth over the open sub-scheme $\mathbf{A}^n \setminus \{(0, \dots, 0)\}$. By the inverse function theorem (see Thm.3.2 and Prop.3.3 in [18]), one has the continuous map

$$q : V_v \setminus \{0\} \rightarrow F_v$$

is open. If $u_0 \notin L_v$, then $L_v + u_0$ is an open and compact subset of $V_v \setminus \{0\}$. Therefore $q(L_v + u_0)$ is an open compact subset of F_v . \square

Corollary 2.3. *Let L be a lattice in V and $u_0 \in V$. If $u_0 \notin L_v$ for $v < \infty_F$, there is an open subgroup U of \mathfrak{o}_v^\times such that $\alpha U \subseteq q(L_v + u_0)$ for all $\alpha \in q(L_v + u_0)$.*

Proof. For any $x \in q(L_v + u_0)$, there is a positive integer a_x such that

$$x + \pi_v^{a_x} \mathfrak{o}_v \subseteq q(L_v + u_0)$$

by Lemma 2.2. By compactness of $q(L_v + u_0)$, one has

$$q(L_v + u_0) = \bigcup_{i=1}^n (x_i + \pi_v^{a_{x_i}} \mathfrak{o}_v).$$

Let

$$a = \max_{1 \leq i \leq n} \{a_{x_i}, a_{x_i} - \text{ord}_v(x_i)\} \quad \text{and} \quad U = 1 + \pi_v^a \mathfrak{o}_v$$

be an open subgroup of \mathfrak{o}_v^\times . For any $\alpha \in q(L_v + u_0)$, there is $1 \leq i_0 \leq n$ such that

$$\alpha \in x_{i_0} + \pi_v^{a_{x_{i_0}}} \mathfrak{o}_v.$$

Therefore

$$\alpha U \subseteq (x_{i_0} + \pi_v^{a_{x_{i_0}}} \mathfrak{o}_v)(1 + \pi_v^a \mathfrak{o}_v) \subseteq x_{i_0} + \pi_v^{a_{x_{i_0}}} \mathfrak{o}_v \subseteq q(L_v + u_0)$$

as required. \square

By Lemma 3.2 in [28] or Lemma 4.2 in [2], one can define the action of $GL_{\mathbb{A}}(V)$ on $L + u_0$.

Definition 2.4. Let L be an \mathfrak{o}_F -lattice and $u_0 \in V$. We define

$gen(L + u_0)$ = the orbit of $(L + u_0)$ under the action of $SO_{\mathbb{A}}(V)$
is called the genus of $(L + u_0)$.

$spn(L + u_0)$ = the orbit of $(L + u_0)$ under the action of $SO(V)\iota(Spin_{\mathbb{A}}(V))$
is called the spinor genus of $(L + u_0)$.

$cls(L + u_0)$ = the orbit of $(L + u_0)$ under the action of $SO(V)$
is called the class of $(L + u_0)$.

It is clear that

$$cls(L + u_0) \subseteq spn(L + u_0) \subseteq gen(L + u_0).$$

Set

$$SO(L + u_0) = \{\sigma \in SO(V) : \sigma(L + u_0) = (L + u_0)\}$$

and

$$SO_{\mathbb{A}}(L + u_0) = \{\sigma \in SO_{\mathbb{A}}(V) : \sigma(L + u_0) = (L + u_0)\}$$

and

$$SO(L_v + u_0) = \{\sigma \in SO(V_v) : \sigma(L_v + u_0) = (L_v + u_0)\}$$

for any $v < \infty_F$.

Proposition 2.5. If $rank(L) \geq 3$, then the number of spinor genera in $gen(L + u_0)$ is given by

$$[\mathbb{I}_F : F^\times \prod_{v \in \Omega_F} \theta(SO(L_v + u_0))]$$

where θ is the map induced by (2.1).

Proof. The number of spinor genera is given by

$$[SO_{\mathbb{A}}(V) : SO(V)\iota(Spin_{\mathbb{A}}(V))SO_{\mathbb{A}}(L + u_0)].$$

Applying the map θ induced by (2.1), one has the bijection

$$SO_{\mathbb{A}}(V)/SO(V)\iota(Spin_{\mathbb{A}}(V))SO_{\mathbb{A}}(L + u_0) \cong \mathbb{I}_F/F^\times \prod_{v \in \Omega_F} \theta(SO(L_v + u_0))$$

by 102:7 in [16]. □

Definition 2.6. Let L_v be a lattice in V_v and $x \in V_v$ for $v < \infty_F$. The coefficient \mathfrak{C}_x of x in L_v is defined as the following fractional ideal

$$\mathfrak{C}_x = \{\alpha \in F_v : \alpha x \in L_v\}.$$

We call x is primitive in L_v if $\mathfrak{C}_x = \mathfrak{o}_v$ and denote $x \in^* L_v$.

If there is $x \in^* L_v$ such that $q(x) = \alpha$, we call α is represented primitively by L_v and denoted by $\alpha \xrightarrow{*} L_v$.

Definition 2.7. Suppose L is an \mathfrak{o}_F -lattice in a quadratic space V and $u_0 \in V$ and $\alpha \in F$. We call α is represented by $\text{gen}(L + u_0)$ (resp. $\text{spn}(L + u_0)$ and $\text{cls}(L + u_0)$) denoted by

$$\alpha \rightarrow \text{gen}(L + u_0) \quad (\text{resp. } \text{spn}(L + u_0) \text{ and } \text{cls}(L + u_0))$$

if there is $(L' + u'_0) \in \text{gen}(L + u_0)$ (resp. $\text{spn}(L + u_0)$ and $\text{cls}(L + u_0)$) such that $\alpha \in q(L' + u'_0)$.

Let T be a finite set of primes of F such that $u_0 \in L_v$ for $v \notin T$. We call α is represented primitively by $\text{gen}(L + u_0)$ (resp. $\text{spn}(L + u_0)$ and $\text{cls}(L + u_0)$) outside T denoted by

$$\alpha \xrightarrow{*}_T \text{gen}(L + u_0) \quad (\text{resp. } \text{spn}(L + u_0) \text{ and } \text{cls}(L + u_0))$$

if there is $(L' + u'_0) \in \text{gen}(L + u_0)$ (resp. $\text{spn}(L + u_0)$ and $\text{cls}(L + u_0)$) and $x \in L' + u'_0$ such that $q(x) = \alpha$ and $x \in {}^*L'_v$ for $v \notin T$.

The following lemma extends some classical results for usual lattices to lattice translations.

Lemma 2.8. Let L be a lattice in V with $\text{rank}(L) \geq 3$ and $u_0 \in V$. Suppose $x \in L + u_0$ with $q(x) = \alpha$. One has the following bijections

$$\begin{aligned} & \{\text{spn}(L' + u'_0) \in \text{gen}(L + u_0) : \alpha \rightarrow \text{spn}(L' + u'_0)\} \\ & \xrightarrow{\cong} X_{\mathbb{A}}(x, L + u_0)SO(V)\iota(\text{Spin}_{\mathbb{A}}(V))/SO(V)\iota(\text{Spin}_{\mathbb{A}}(V))SO_{\mathbb{A}}(L + u_0) \\ & \xrightarrow{\cong} F^{\times}\theta(X_{\mathbb{A}}(x, L + u_0))/F^{\times} \prod_{v \in \Omega_F} \theta(SO(L_v + u_0)) \end{aligned}$$

where

$$X_{\mathbb{A}}(x, L + u_0) = \{\sigma_{\mathbb{A}} \in SO_{\mathbb{A}}(V) : x \in \sigma_{\mathbb{A}} \circ (L + u_0)\}.$$

Moreover, if T is a finite set of primes of F such that $u_0 \in L_v$ for $v \notin T$, then one has the following bijections

$$\begin{aligned} & \{\text{spn}(L' + u'_0) \in \text{gen}(L + u_0) : \alpha \xrightarrow{*}_T \text{spn}(L' + u'_0)\} \\ & \xrightarrow{\cong} X_T^*(x, L + u_0)SO(V)\iota(\text{Spin}_{\mathbb{A}}(V))/SO(V)\iota(\text{Spin}_{\mathbb{A}}(V))SO_{\mathbb{A}}(L + u_0) \\ & \xrightarrow{\cong} F^{\times}\theta(X_T^*(x, L + u_0))/F^{\times} \prod_{v \in \Omega_F} \theta(SO(L_v + u_0)) \end{aligned}$$

where

$$X_T^*(x, L + u_0) = \{(\sigma_v) \in SO_{\mathbb{A}}(V) : x \in (\sigma_v) \circ (L + u_0) \text{ and } x \in {}^*\sigma_v L_v \text{ for } v \notin T\}.$$

Proof. If $\alpha \rightarrow \text{spn}(L' + u'_0)$, there are $\sigma \in SO(V)$ and $\tau_{\mathbb{A}} \in \iota(\text{Spin}_{\mathbb{A}}(V))$ such that

$$x \in \sigma\tau_{\mathbb{A}} \circ (L' + u'_0)$$

by Witt Theorem (see [16] §42 F). Since $\text{spn}(L' + u'_0) \in \text{gen}(L + u_0)$, one has $\varrho_{\mathbb{A}} \in SO_{\mathbb{A}}(V)$ such that $\varrho_{\mathbb{A}} \circ (L + u_0) = L' + u'_0$. Therefore one has

$$\sigma\tau_{\mathbb{A}}\varrho_{\mathbb{A}} \in X_{\mathbb{A}}(x, L + u_0)$$

and defines the first map by sending

$$\text{spn}(L' + u'_0) \mapsto \sigma\tau_{\mathbb{A}}\varrho_{\mathbb{A}}SO(V)\iota(\text{Spin}_{\mathbb{A}}(V))SO_{\mathbb{A}}(L + u_0).$$

It is straight forward to verify that the above map is well-defined and the inverse map is given by sending

$$\sigma_{\mathbb{A}}SO(V)\iota(Spin_{\mathbb{A}}(V))SO_{\mathbb{A}}(L + u_0) \mapsto spn(\sigma_{\mathbb{A}} \circ (L + u_0)).$$

Applying θ in (2.1), one obtains the second bijection by 102:7 in [16].

If $\alpha \xrightarrow{*}_T spn(L' + u'_0)$, then $x \in^* \sigma\tau_v(L'_v)$ for $v \notin T$ in the above argument. Therefore

$$\sigma\tau_{\mathbb{A}}\varrho_{\mathbb{A}} \in X_T^*(x, L + u_0)$$

and the rest of the proof follows from the exact same argument as above. \square

Corollary 2.9. *Let L be a lattice in V and $u_0 \in V$ and $\alpha \in F^\times$. Suppose one of the following conditions holds*

- i) $rank(L) \geq 4$; or $rank(L) = 3$ and $-\alpha \cdot \det(L) \in (F^\times)^2$.*
- ii) $rank(L) = 3$ and there is a non-dyadic finite prime v_0 such that L_{v_0} is unimodular and $u_0 \in L_{v_0}$ with $ord_{v_0}(\alpha) > 0$ and $-\alpha \cdot \det(L) \notin (F_{v_0}^\times)^2$.*

If $\alpha \rightarrow gen(L + u_0)$, then every spinor genus in $gen(L + u_0)$ represents α . Moreover, if T is the finite set of primes of T such that $u_0 \in L_v$ for $v \notin T$ and $v_0 \in T$ in case ii) and $\alpha \xrightarrow{}_T gen(L + u_0)$, then every spinor genus in $gen(L + u_0)$ represents α primitively outside T .*

Proof. Since $\alpha \rightarrow gen(L + u_0)$, there are $(K + w_0) \in gen(L + u_0)$ and $x \in (K + w_0)$ such that $q(x) = \alpha$. Let $V = Fx \perp W$. Then

$$SO_{\mathbb{A}}(W) \subseteq X_{\mathbb{A}}(x, K + w_0) \quad \text{and} \quad [\mathbb{I}_F : F^\times \theta(SO_{\mathbb{A}}(W))] \leq 2$$

by Lemma 2.8 and 91:6 in [16]. Moreover

$$[\mathbb{I}_F : F^\times \theta(SO_{\mathbb{A}}(W))] = 2 \quad \text{if and only if} \quad -\det(W) \notin (F^\times)^2 \quad \text{and} \quad \dim(W) = 2$$

with

$$\mathbb{I}_F / F^\times \theta(SO_{\mathbb{A}}(W)) \cong Gal(F(\sqrt{-\det(W)})/F)$$

by the Artin map.

If condition i) holds, one already has

$$\mathbb{I}_F = F^\times \theta(SO_{\mathbb{A}}(W)) = F^\times \theta(X_{\mathbb{A}}(x, K + w_0))$$

and every spinor genera in $gen(L + u_0)$ represents α by Lemma 2.8 and Prop.2.5.

If condition ii) holds, one has

$$F^\times \theta(SO_{\mathbb{A}}(W)) \neq F^\times \theta(X_{\mathbb{A}}(x, K + w_0)) \quad \text{if and only if} \quad \theta(SO_{\mathbb{A}}(W)) \neq \theta(X_{\mathbb{A}}(x, K + w_0))$$

by (5.6) Proposition of Chapter VI in [15]. By Theorem 5.1 in [9] at v_0 , one has

$$\theta(X_{v_0}(x, K + w_0)) \neq \theta(SO(W_{v_0}))$$

where $X_{v_0}(x, K + w_0)$ is the v_0 -component of $X_{\mathbb{A}}(x, K + w_0)$. Therefore one still has

$$\mathbb{I}_F = F^\times \theta(X_{\mathbb{A}}(x, K + w_0))$$

and every spinor genera in $gen(L + u_0)$ represents α by Lemma 2.8 and Prop.2.5.

If $\alpha \xrightarrow{*}_T gen(L + u_0)$, one can further assume that $x \in^* K_v$ for $v \notin T$. Therefore

$$SO_{\mathbb{A}}(W) \subseteq X_T^*(x, K + w_0)$$

and the result follows from the same argument as above by replacing $X_{\mathbb{A}}(x, K + w_0)$ with $X_T^*(x, K + w_0)$. \square

The following lemma extends some ideas for usual lattices in [29] to lattice translations.

Lemma 2.10. *Let L be a lattice with $\text{rank}(L) = 3$ in V and $u_0 \in V$. Suppose v_0 is a finite non-dyadic prime such that L_{v_0} is unimodular with $u_0 \in L_{v_0}$ and $t \in \mathfrak{o}_F$ such that*

$$\text{ord}_v(t) = 0 \quad \text{and} \quad (t - 1)u_0 \in L_v$$

for all finite primes v except $v = v_0$ and $-\alpha \cdot \det(L) \in (F_{v_0}^\times)^2$. If $\alpha \rightarrow \text{spn}(L + u_0)$ and $t^{-2k}\alpha \rightarrow \text{gen}(L + u_0)$ for a positive integer k , then $t^{-2k}\alpha \rightarrow \text{spn}(L + u_0)$.

Moreover, if T is a finite set of primes of F containing v_0 such that $u_0 \in L_v$ for $v \notin T$ and $\alpha \xrightarrow{}_T \text{spn}(L + u_0)$ and $t^{-2k}\alpha \rightarrow \text{gen}(L + u_0)$ for a positive integer k , then $t^{-2k}\alpha \xrightarrow{*}_T \text{spn}(L + u_0)$.*

Proof. By Corollary 2.9, one only needs to consider $-\alpha \cdot \det(L) \notin (F^\times)^2$. Without loss of generality, we can assume that there is $x \in L + u_0$ such that $q(x) = \alpha$. Let $y = t^{-k}x$. Since $t^{-2k}\alpha \rightarrow \text{gen}(L + u_0)$, one has $\sigma_{v_0} \in SO(V_{v_0})$ such that $y \in \sigma_{v_0}(L_{v_0} + u_0)$. Define

$$\sigma_v = \begin{cases} \sigma_{v_0} & \text{if } v = v_0 \\ 1 & \text{otherwise} \end{cases}$$

and $K + w_0 = (\sigma_v)_{v \in \Omega_F} \circ (L + u_0)$. Then $y \in K + w_0$ and the spinor genera in $\text{gen}(L + u_0)$ which represent $t^{-2k}\alpha$ are given by

$$F^\times \theta(X_{\mathbb{A}}(y, K + w_0)) / F^\times \prod_{v \in \Omega_F} \theta(SO(K_v + w_0))$$

by Lemma 2.8. Let $E = F(\sqrt{-\alpha \cdot \det(L)})$. Since v_0 splits completely in E/F , one has

$$\theta((\sigma_v)_{v \in \Omega_F}^{-1}) \in N_{E/F}(\mathbb{I}_E) \subseteq \theta(X_{\mathbb{A}}(y, K + w_0)).$$

This implies that

$$(\sigma_v)_{v \in \Omega_F}^{-1} SO(V) \iota(\text{Spin}_{\mathbb{A}}(V)) SO_{\mathbb{A}}(K + w_0) \in X_{\mathbb{A}}(y, K + w_0) SO(V) \iota(\text{Spin}_{\mathbb{A}}(V))$$

and

$$\text{spn}((\sigma_v)_{v \in \Omega_F}^{-1} \circ (K + w_0)) = \text{spn}(L + u_0)$$

represents $t^{-2k}\alpha$ by Lemma 2.8.

If $\alpha \xrightarrow{*}_T \text{spn}(L + u_0)$, we can further assume that $x \in^* L_v$ for $v \notin T$ which implies that $y \in^* K_v$ for $v \notin T$ in the above argument. The result follows from the same argument as above by replacing $X_{\mathbb{A}}(x, K + w_0)$ with $X_T^*(x, K + w_0)$. \square

The following proposition is based on Kneser's arithmetic isotropic idea. Several variants have already appeared, for example Lemma 1.2 in [8].

Proposition 2.11. *Let L be a lattice in V with $\dim(V) \geq 3$ and $u_0 \in V$. Suppose v_0 is a non-dyadic finite prime such that L_{v_0} is unimodular and $u_0 \in L_{v_0}$. Then there is a positive number $h = h(L, u_0, v_0)$ depending only on L, u_0 and v_0 such that*

i) if $\alpha \rightarrow \text{spn}(L + u_0)$ and $\text{ord}_{v_0}(\alpha) \geq h$ with $\alpha \in F^\times$, one has $\alpha \rightarrow \text{cls}(L + u_0)$.

ii) if T is a finite set of primes of F containing v_0 and satisfying $u_0 \in L_v$ for $v \notin T$ and $\alpha \xrightarrow{*}_T \text{spn}(L + u_0)$ and $\text{ord}_{v_0}(\alpha) \geq h$ with $\alpha \in F^\times$, one has $\alpha \xrightarrow{*}_T \text{cls}(L + u_0)$.

Proof. Since the class number of \mathfrak{o}_F is finite, the suitable power of the prime v_0 is principal ideal. Then there is $t_0 \in \mathfrak{o}_F$ such that $\text{ord}_{v_0}(t_0) > 0$ and $\text{ord}_v(t_0) = 0$ for all finite $v \neq v_0$.

Let $T_1 = \{v < \infty_F : u_0 \notin L_v\}$. For any $v \in T_1$, there is an open subgroup U_v of \mathfrak{o}_v^\times such that $\alpha U_v \subseteq q(L_v + u_0)$ for all $\alpha \in q(L_v + u_0)$ by Corollary 2.3. Moreover, one can shrink U_v if necessary such that $(1 - \eta)u_0 \in L_v$ for all $\eta \in U_v$. Let

$$h_0 = \prod_{v \in T_1} [o_v^\times : U_v] \quad \text{and} \quad t_1 = t_0^{h_0}.$$

Then $t_1 \in U_v$ for all $v \in T_1$.

Fix the representative $L + u_0, L_1 + u_1, \dots, L_s + u_s$ of all classes in $\text{spn}(L + u_0)$. The strong approximation for the spin group off $S = \{\infty_F, v_0\}$ implies

$$\text{cls}(L \otimes_{\mathfrak{o}_F} \mathfrak{o}_S + u_0) = \text{cls}(L_i \otimes_{\mathfrak{o}_F} \mathfrak{o}_S + u_i)$$

for $1 \leq i \leq s$ by the condition that V_{v_0} is isotropic. There is $\sigma_i \in SO(V)$ such that

$$L_i \otimes_{\mathfrak{o}_F} \mathfrak{o}_S = \sigma_i(L \otimes_{\mathfrak{o}_F} \mathfrak{o}_S) \quad \text{and} \quad u_i - \sigma_i(u_0) \in \sigma_i(L \otimes_{\mathfrak{o}_F} \mathfrak{o}_S)$$

for $1 \leq i \leq s$. Therefore there is a positive integer l_i such that

$$t_1^{l_i} L_i \subseteq \sigma_i(L) \quad \text{and} \quad t_1^{l_i} (u_i - \sigma_i(u_0)) \in \sigma_i(L)$$

for $1 \leq i \leq s$. Let

$$h_1 = \max_{1 \leq i \leq s} \{l_i\} \quad \text{and} \quad h = 2h_1 \cdot \text{ord}_{v_0}(t_1) + 1$$

which only depends on L, u_0 and v_0 .

If $\alpha \rightarrow \text{spn}(L + u_0)$ and $\text{ord}_{v_0}(\alpha) \geq h$ for $\alpha \in F^\times$, then

$$\beta = t_1^{-2h_1} \alpha \rightarrow \text{gen}(L + u_0)$$

by Corollary 2.3 for $v \in T_1$ and [17] for $v = v_0$. We further have that $\beta \rightarrow \text{spn}(L + u_0)$ by Corollary 2.9 and Lemma 2.10. There is $0 \leq i \leq s$ such that $\beta \in q(L_i + u_i)$. Therefore

$$\alpha = \beta \cdot t_1^{2h_1} \in t_1^{2h_1} q(L_i + u_i) = q(t_1^{h_1} L_i + t_1^{h_1} u_i) \subseteq q(L + t_1^{h_1} u_0)$$

and i) follows from the fact $(t_1^{h_1} - 1)u_0 \in L$ by the choice of U_v for $v \in T_1$.

If $\alpha \xrightarrow{*}_T \text{spn}(L + u_0)$ and $\text{ord}_{v_0}(\alpha) \geq h$ for $\alpha \in F^\times$, then

$$T_1 \cup \{v_0\} \subseteq T \quad \text{and} \quad \beta = t_1^{-2h_1} \alpha \xrightarrow{*}_T \text{gen}(L + u_0)$$

by Corollary 2.3 for $v \in T_1$ and [17] for $v = v_0$. By Corollary 2.9 and Lemma 2.10, one has $\beta \xrightarrow{*}_T \text{spn}(L + u_0)$. There is $0 \leq i \leq s$ and $x \in L_i + u_i$ with $x \in^* (L_i)_v$ for $v \notin T$ such that $q(x) = \beta$. Therefore one has $\alpha = q(t_1^{h_1} x)$ with

$$t_1^{h_1} x \in \sigma_i(L + u_0) \quad \text{and} \quad t_1^{h_1} x \in^* \sigma_i L_v$$

for all $v \notin T$ and ii) follows. □

3. STRONG APPROXIMATION WITH BRAUER-MANIN OBSTRUCTION

Let X be the affine variety over F defined by the equation

$$q(x_1, \dots, x_n) = p(t) \quad (3.1)$$

where $q(x_1, \dots, x_n)$ is a non-degenerate quadratic form over F with $n \geq 3$ and $p(t)$ is a non-zero polynomial over F . Let $f : \tilde{X} \rightarrow X$ be a resolution of singularities for X , i.e. \tilde{X} is a smooth, geometrically integral F -variety and f is a proper and birational F -morphism such that

$$f : f^{-1}(X_{\text{smooth}}) \xrightarrow{\cong} X_{\text{smooth}}$$

is isomorphism, where X_{smooth} is the smooth locus of X .

Definition 3.2. A sign s_v for a real place v of F is defined to be 1 or -1 . An element $\alpha \in F^\times$ is called to be with sign s_v at v if $s_v \cdot \alpha > 0$ in F_v .

The following lemma is well-known and is regarded as the generalized Dirichlet arithmetic progression theorem over number fields, which is the immediate consequence of the Chebatorev density theorem and the class field theory.

Lemma 3.3. Let T be a finite set of non-archimedean primes of F and $\alpha_v \in F_v$ for each $v \in T$. Fix a sign s_v for each real place $v \in \infty_F$. For any $\epsilon > 0$, there are $\alpha \in F^\times$ and a non-archimedean prime v_0 outside T of degree one such that

- 1) $|\alpha - \alpha_v|_v < \epsilon$ for all $v \in T$.
- 2) α is unit at any non-archimedean prime $v \notin T \cup \{v_0\}$ and $\text{ord}_{v_0}(\alpha) = 1$.
- 3) $s_v \cdot \alpha > 0$ in F_v .

Proof. Without loss of generality, we can assume that $a_v \neq 0$ for $v \in T$ by replacing a_v with $a'_v \in F_v^\times$ and $|a'_v|_v < \epsilon$ if necessary. Let \mathbb{R}^+ be the set of all positive reals. Choose a positive integer c_v such that

$$|\pi_v^{c_v}|_v < |a_v|_v^{-1} \epsilon$$

for any $v \in T$. By the class field theory, there is an abelian extension E/F such that

$$\mathbb{I}_F / (F^\times \cdot \prod_{v \in \Omega_F} U_v) \cong \text{Gal}(E/F)$$

by Artin reciprocity law, where

$$U_v = \begin{cases} \mathfrak{o}_v^\times & v \notin T \cup \infty_F \\ 1 + \pi_v^{c_v} \mathfrak{o}_v & v \in T \\ F_v^\times & v \text{ complex} \\ \mathbb{R}^+ & v \text{ real.} \end{cases}$$

Let

$$i_v = \begin{cases} 1 & v \notin T \cup \infty_F \\ a_v^{-1} & v \in T \\ 1 & v \text{ complex} \\ s_v & v \text{ real.} \end{cases}$$

Then $(i_v)_{v \in \Omega_F} \in \mathbb{I}_F$ and gives an element σ in $\text{Gal}(E/F)$. By the Chebatorev density theorem, there is a finite prime v_0 of degree one outside T such that the Frobenius of v_0 is σ . Therefore one has

$$(j_v)_{v \in \Omega_F} \cdot (i_v)_{v \in \Omega_F}^{-1} \in F^\times \cdot \prod_{v \in \Omega_F} U_v$$

where

$$j_v = \begin{cases} \pi_{v_0} & v = v_0 \\ 1 & \text{otherwise} \end{cases}$$

for all $v \in \Omega_F$. There is $\alpha \in F^\times$ such that

$$\alpha^{-1} \cdot (j_v)_{v \in \Omega_F} \cdot (i_v)_{v \in \Omega_F}^{-1} \in \prod_{v \in \Omega_F} U_v$$

as required. \square

Corollary 3.4. *Let T be a finite set of non-archimedean primes of F and $\alpha_v \in F_v$ for each $v \in T$. Fix a sign s_v for each real place $v \in \infty_F$. For any sufficiently small $\epsilon > 0$ and any sufficiently large $C > 0$, there are $\alpha \in F$ such that*

- 1) $|\alpha - \alpha_v|_v < \epsilon$ for all $v \in T$.
- 2) $\alpha \in \mathfrak{o}_v$ for all $v \notin T \cup \infty_F$.
- 3) $s_v \cdot \alpha > C$ at each real place v of F .

Proof. By Lemma 3.3, there are α_1 and α_2 in F^\times such that

$$\begin{cases} |\alpha_1 - \alpha_v|_v < \epsilon \text{ and } |\alpha_2|_v < \epsilon & \text{for all } v \in T \\ s_v \cdot \alpha_1 > 0 \text{ and } s_v \cdot \alpha_2 > 0 & \text{for all real places } v \\ \alpha_1 \in \mathfrak{o}_v \text{ and } \alpha_2 \in \mathfrak{o}_v & \text{for all } v \notin T \cup \infty_F. \end{cases}$$

Let k be a positive integer such that $s_v \cdot (\alpha_1 + k\alpha_2) > C$ for all real places v of F . Then $\alpha = \alpha_1 + k\alpha_2$ is as required. \square

The main result of this section is the following theorem.

Theorem 3.5. *If $\tilde{X}(F_{\infty_F})$ is not compact, then strong approximation with Brauer-Manin obstruction off ∞_F holds for \tilde{X} .*

Proof. By Proposition 3.4 and Theorem 6.4 in [5], one only needs to consider the case that F is totally real and q is definite over F_v for all $v \in \infty_F$. Let c be the leading coefficient of $p(t)$.

Fix $\xi \in \text{Br}(\tilde{X})$ such that ξ gives the generator of $\text{Br}(\tilde{X})/\text{Br}(F)$ by Proposition 5.6 in [5]. Let V be a quadratic space over F defined by q . If W is an open subset of $\tilde{X}(\mathbb{A}_F)$ such that

$$\tilde{X}(\mathbb{A}_F)^{\text{Br}(\tilde{X})} \cap W \neq \emptyset,$$

there is a finite subset T of Ω_F containing ∞_F and all dyadic primes in F such that

- 1). T contains all denominators of coefficients of $q(x_1, \dots, x_n)$ and $p(t)$.
- 2). (3.1) defines an \mathfrak{o}_T -model of X and a proper morphism of \mathfrak{o}_T -model $f : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$ extends the morphism $f : \tilde{X} \rightarrow X$.

- 3). ξ takes the trivial value over $\tilde{\mathbf{X}}(\mathfrak{o}_v)$ for all $v \notin T$.
 4). there is an non-empty open subset U_v of $X(F_v)_{cent}$ for each $v \in \Omega_F$ satisfying

$$f^{-1}\left(\prod_{v \in \Omega_F} U_v\right) \subseteq W \quad \text{and} \quad \tilde{X}(\mathbb{A}_F)^{Br(\tilde{X})} \cap f^{-1}\left(\prod_{v \in \Omega_F} U_v\right) \neq \emptyset.$$

Moreover, one can choose U_v as follows

$$U_v = \{(x_v, t_v) \in X(F_v) : x_v \in L_v + u_0 \text{ and } |t_v - b_v|_v < \epsilon\}$$

for all $v \in T \setminus \infty_F$ with sufficiently small $\epsilon > 0$, where L is a lattice in V and $u_0 \in V$ and $b_v \in F_v$ with $p(b_v) \neq 0$ and $u_0 \in L_v$ for all $v \notin T$ by Chinese Remainder Theorem and

$$U_v = \mathbf{X}(\mathfrak{o}_v) \cap X(F_v)_{cent} = \{(x_v, t_v) \in X(F_v)_{cent} : x_v \in L_v \text{ and } t_v \in \mathfrak{o}_v\}$$

for all $v \notin T$ by Lemma 8.1 and Lemma 8.3 in [5], where $X(F_v)_{cent}$ is the closure of $X_{smooth}(F_v)$ in $X(F_v)$. By increasing T if necessary, we also can assume that $ord_v(det(L)) = 0$ for all $v \notin T$.

Let

$$P \in \tilde{X}(\mathbb{A}_F)^{Br(\tilde{X})} \cap f^{-1}\left(\prod_{v \in \Omega_F} U_v\right) \quad \text{and} \quad f(P) = (a_v, b_v)_{v \in \Omega_F} \in X(\mathbb{A}_F) \cap \prod_{v \in \Omega_F} X(F_v)_{cent}.$$

By the Chebatorev density theorem, there is $v_1 \notin T$ such that $p(t)$ has a root in F_{v_1} . Choose $t_{v_1} \in \mathfrak{o}_{v_1}$ such that $p(t_{v_1}) \neq 0$ and $ord_{v_1}(p(t_{v_1})) > h$ where $h = h(L, u_0, T, v_1)$ is the positive number given in Proposition 2.11.

If $deg(p(t))$ is odd, one can choose the sign s_v for all real places $v \in \infty_F$ as follows

$$s_v = \begin{cases} 1 & \text{if } c \cdot q \text{ is positive definite at } v \\ -1 & \text{otherwise} \end{cases}$$

and the sufficiently large constant $C > 0$ such that $p(t)$ and $c \cdot t$ have the same sign in F_v if $|t|_v > C$ for all $v \in \infty_F$. For sufficiently small $\epsilon > 0$, there is $t_0 \in F$ such that

$$\begin{cases} |t_0 - b_v|_v < \epsilon & v \in T \setminus \infty_F \\ |t_0 - t_{v_1}|_{v_1} < \epsilon & v = v_1 \\ t_0 \in \mathfrak{o}_v & v \notin T \cup \{v_1\} \\ s_v \cdot t_0 > C & v \in \infty_F \end{cases}$$

by Corollary 3.4. By Lemma 4.2 in [5] for $v \notin T$ and the inverse function theorem (see Theorem 3.2 in [18]) for $v \in T \setminus \infty_F$ and the choice of sign function s_v for $v \in \infty_F$, one obtains

$$p(t_0) \rightarrow gen(L + u_0).$$

Otherwise $deg(p(t))$ is even. Since $X(F_{\infty_F})$ is not compact, there is $v_0 \in \infty_F$ such that $X(F_{v_0})$ is not compact. Choose a sufficiently large constant $C > 0$ such that $c \cdot p(t) > 0$ in F_{v_0} if $|t|_{v_0} > C$. Choose $v_2 \notin T \cup \{v_1\}$ and $t_{v_2} \in \mathfrak{o}_{v_2}$ with $t_{v_2} \neq 0$ such that

$$|t_{v_2}|_{v_2} < C^{-1}|t_{v_1}|_{v_1}^{-1} \prod_{v \in T \setminus \{v_0\}} (|b_v|_v + 1)^{-1}.$$

Applying Chinese Remainder Theorem, one obtains $t_0 \in F$ such that

$$\begin{cases} |t_0 - b_v|_v < \epsilon & v \in T \setminus \{v_0\} \\ |t_0 - t_v|_v < \epsilon & v \in \{v_1, v_2\} \\ t_0 \in \mathfrak{o}_v & v \notin T \cup \{v_1, v_2\}. \end{cases}$$

By the product formula, one has

$$1 \leq |t_0|_{v_0} \cdot |t_0|_{v_1} \cdot |t_0|_{v_2} \cdot \prod_{v \in T \setminus \{v_0\}} |t_0|_v < |t_0|_{v_0} \cdot C^{-1}.$$

This implies that $c \cdot p(t_0) > 0$ in F_{v_0} . Since $X(F_{v_0})$ is not compact, one concludes that $c \cdot q$ is positive definite at v_0 . Therefore one has

$$p(t_0) \rightarrow \text{gen}(L + u_0)$$

by Lemma 4.2 in [5] for $v \notin T$ and the inverse function theorem (see Theorem 3.2 in [18]) for $v \in T \setminus \{v_0\}$ and the above property at v_0 .

Moreover, one has

$$p(t_0) \rightarrow \text{spn}(L + u_0)$$

by Proposition 5.6 (c) and Lemma 4.4 in [5] and the functoriality and continuity of Brauer-Manin pairing (see (5.3) in [24]) and the proof Proposition 7.3 in [4]. Therefore

$$p(t_0) \rightarrow \text{cls}(L + u_0)$$

by Proposition 2.11 i). Since all points in the fiber of X over $t = t_0$ are smooth, the morphism f induces

$$\tilde{X}_{t_0} \cong X_{t_0}$$

defined by $q(x_1, \dots, x_n) = p(t_0)$. This implies that

$$\tilde{X}_{t_0}(F) \cap f^{-1}\left(\prod_{v \in \Omega_F} U_v\right) \neq \emptyset$$

and the proof is complete. □

Remark 3.6. *The D defined by (2.2) in [27] is equal to $p(t)$ up to a constant in \mathbb{Q}^\times . When $n \geq 4$, one has $\text{Br}(\tilde{X}) = \text{Br}(F)$ and strong approximation off ∞_F holds for \tilde{X} or by the same line of the proof of Theorem 3.5. Therefore the local-global principle holds for (1.1) with $n \geq 4$, which is equivalent to Theorem 1 and Theorem 2 in [27]. The most interesting case is that $n = 3$. The explicit computation for $\text{Br}(\tilde{X})/\text{Br}(F)$ is given by Proposition 5.6 in [5]. The condition of Theorem 3 in [27] is that $p(t)$ has a root over \mathbb{Q}_p for almost all primes p . This condition implies that $\text{Br}(\tilde{X}) = \text{Br}(F)$ by Remark 6.6 in [5]. Therefore strong approximation off ∞_F holds for \tilde{X} with $n = 3$ under such a condition. This implies that the local-global principle holds for (1.1) with $n = 3$ under such a condition, which is equivalent to Theorem 3 in [27].*

4. REPRESENTATIONS OF DEFINITE QUADRATIC POLYNOMIALS

By using the circle method as pointed out in [26], one can prove the representability of definite quadratic polynomials with more than three variables. On the other hand, primitive representations of definite quadratic forms of more than four variables with congruence conditions is proved in Theorem 2.1 of [7] by an arithmetic method. In this section, we will point out that these two statements are equivalent to each other and extend them over number fields based on the results in §2.

Definition 4.1. *Let V be a non-degenerated quadratic space over F and $w \in \infty_F$ such that V_w is positive definite and T be a finite subset containing all dyadic primes with $T \subset (\Omega_F \setminus \infty_F)$.*

Statement (CC) with respect to T : *If L is a lattice in V satisfying that L_v is unimodular for all $v \notin T$ and s is a positive integer, there is a constant $c = c(L, T, s)$ depending only on L , T and s such that for any $\alpha \in F$ satisfying*

- 1) $\alpha \rightarrow \text{gen}(L)$
- 2) $x_v \in L_v$ with $q(x_v) = \alpha$ for $v \in T$ and $x_v \in {}^*L_v$ when V_v is anisotropic with $v \in T$
- 3) $\alpha > c$ at w

one has $x \in L$ with $q(x) = \alpha$ such that

$$\begin{cases} x \equiv x_v \pmod{\pi_v^s L_v} & \text{for } v \in T \\ x \in {}^*L_v & \text{for } v \notin T \end{cases}$$

Statement (LT) with respect to T : *If L is a lattice in V and $u_0 \in V$ satisfying that L_v is unimodular and $u_0 \in L_v$ for all $v \notin T$, there is a constant $c = c(L, T, u_0)$ depending only on L , T and u_0 such that for any $\alpha \in F$ satisfying*

- 1) $\alpha \rightarrow \text{gen}(L + u_0)$
- 2) $\alpha \xrightarrow{*} L_v$ when V_v is anisotropic and $u_0 \in L_v$ for $v \in T$
- 3) $\alpha > c$ at w

*one has $x \in L + u_0$ with $x \in {}^*L_v$ for $v \notin T$ such that $q(x) = \alpha$.*

Proposition 4.2. *Let V be a non-degenerated quadratic space over F and $w \in \infty_F$ such that V_w is positive definite and T be a finite subset with $T \subset (\Omega_F \setminus \infty_F)$. Then Statement (CC) with respect to T holds if and only if Statement (LT) with respect to T holds.*

Proof. (\Rightarrow) Let $K = L + \mathfrak{o}_F u_0$ be a lattice in V . Since $\alpha \rightarrow \text{gen}(L + u_0)$, there is $x_v \in L_v$ such that $q(x_v + u_0) = \alpha$ for $v \in T$. Choose a positive integer s so large such that $\pi_v^s u_0 \in L_v$ for all $v \in T$. By Statement (CC) with respect to T , there is $x \in K$ with $q(x) = \alpha$ such that $x \equiv x_v + u_0 \pmod{\pi_v^s K_v}$ for $v \in T$ and $x \in {}^*K_v$ for $v \notin T$. Since $\pi_v^s K_v \subseteq L_v$ for all $v \in T$ and $K_v = L_v$ for all $v \notin T$, one concludes that $x \in L + u_0$ as required.

(\Leftarrow) Conversely, define a lattice K in V

$$\begin{cases} K_v = \pi_v^s L_v & v \in T \\ K_v = L_v & v \notin T \end{cases}$$

and $u_0 \in V$ satisfying

$$\begin{cases} u_0 - x_v \in \pi_v^s L_v & v \in T \\ u_0 \in L_v & v \notin T \end{cases}$$

by Chinese Remainder Theorem. By Statement (LT) with respect to T , there is $x \in K + u_0$ with $x \in {}^* K_v$ for $v \notin T$ such that $q(x) = \alpha$. Then

$$\begin{cases} x \equiv u_0 \equiv x_v \pmod{\pi_v^s L_v} & \text{for } v \in T \\ x \in {}^* L_v & \text{for } v \notin T \end{cases}$$

by $K_v = L_v$ for $v \notin T$. □

By applying Lemma 3.3, one can obtain the following approximation result which was first proved by Kitaoka for usual lattices over \mathbb{Z} in [12] (see also Lemma 1.6 in [8] and Theorem 6.2.1 in [13]).

Lemma 4.3. *Suppose V is a non-degenerated quadratic space over F such that $\dim(V) \geq 3$ or $\dim(V) = 2$ with $-\det(V) \notin (F^\times)^2$. Let L be a lattice in V and $u_0 \in V$ and T be a finite set of non-archimedean primes of F such that T contains all dyadic primes and L_v is unimodular and $u_0 \in L_v$ for $v \notin T$. Take $u_v \in L_v + u_0$ for each $v \in T$. For any $\epsilon > 0$, there is $u \in L + u_0$ such that*

- 1) $|u - u_v|_v < \epsilon$ for all $v \in T$
- 2) there is a finite prime v_0 outside T satisfying that $q(u)$ is unit at any non-archimedean prime $v \notin T \cup \{v_0\}$ and $\text{ord}_{v_0}(q(u)) = 1$.

Proof. Step 1. If $\dim(V) = 2$ and $-\det(V) \notin (F^\times)^2$, there is $a \in F^\times$ such that

$$\phi : (V/F, a \cdot q) \cong (E/F, N_{E/F})$$

as quadratic spaces, where $E = F(\sqrt{-\det(V)})$ and $N_{E/F}$ is the usual norm map. Let

$$T_1 = \{v \in \Omega_F \setminus (T \cup \infty_F) : \phi(L)_v \neq o_{Ev}\}$$

where o_E is the integral closure of \mathfrak{o}_F inside E/F and $u_v \in L_v$ such that $q(u_v) \in o_v^\times$ for all $v \in T_1$ since L_v is unimodular for $v \notin T$.

For sufficiently small $\epsilon > 0$, one obtains $\lambda \in E$ and a non-archimedean prime w_0 of E of degree one such that $|\lambda - \phi(u_v)|_w < \epsilon$ for all w above $T \cup T_1$ and λ is unit at any non-archimedean prime w which is not above $T \cup T_1$ but $\text{ord}_{w_0}(\lambda) = 1$ by applying Lemma 3.3 for the set of primes of E above $T \cup T_1$.

We claim that $u = \phi^{-1}(\lambda)$ is as required. Indeed, $u \in L_v + u_0$ for all $v \in \Omega_F$ by our construction. This implies that $u \in L + u_0$. For any finite prime v with $v \notin T \cup T_1$, one has both $q(L_v)$ and $N_{E/F}(\mathfrak{o}_{Ev})$ contain the units of \mathfrak{o}_v^\times . This implies that $\text{ord}_v(a) = 0$ for $v \notin T \cup T_1$. Therefore

$$\text{ord}_{v_0}(q(u)) = \text{ord}_{v_0}(a \cdot q(u)) = \text{ord}_{v_0}(N_{E/F}(\lambda)) = 1$$

where v_0 is the prime in F below w_0 and

$$\text{ord}_v(q(u)) = \text{ord}_v(a \cdot q(u)) = \text{ord}_v(N_{E/F}(\lambda)) = 0$$

for all finite primes $v \notin (T \cup T_1 \cup \{v_0\})$. Since $|u - u_v|_v < \epsilon$ with sufficient small ϵ for $v \in T_1$, one has $\text{ord}_v(q(u)) = \text{ord}_v(q(u_v)) = 0$ for $v \in T_1$ by the choice of u_v for $v \in T_1$. The claim follows.

Step 2. If $\dim(V) \geq 3$, we set $M = L + \mathfrak{o}_F u_0$. Choose a finite $v_1 \notin T$. Then

$$M_{v_1} = L_{v_1}, \quad \mathfrak{o}_{v_1}^\times \subset q(M_{v_1}) \quad \text{and} \quad [\mathfrak{o}_{v_1}^\times : (\mathfrak{o}_{v_1}^\times)^2] = 2.$$

There is a unimodular sublattice $K \subset M_{v_1}$ satisfying

$$\text{rank}(K) = 2 \quad \text{and} \quad -\det(K) \notin (F_{v_1}^\times)^2.$$

Write $K = \mathfrak{o}_{v_1} u_{v_1} \perp \mathfrak{o}_{v_1} \eta_{v_1}$. For sufficiently small $\epsilon > 0$, there is $\xi \in M$ such that

$$|\xi - u_{v_1}|_{v_1} < \epsilon \quad \text{and} \quad |\xi - u_v|_v < \epsilon$$

for $v \in T$ by Chinese Remainder Theorem. Let

$$T_1 = \{v \in \Omega_F \setminus (T \cup \{v_1\} \cup \infty_F) : \text{ord}_v(q(\xi)) \neq 0\}.$$

Choose $\eta_v \in M_v$ such that $\text{ord}_v(q(\eta_v)) = 0$ for each $v \in T_1$. For sufficiently small $\epsilon > 0$, there is $\eta \in M$ such that

$$|\eta - \eta_{v_1}|_{v_1} < \epsilon \quad \text{and} \quad |\eta - \eta_v|_v < \epsilon$$

for $v \in T_1$ and ξ and η are linearly independent over F by Chinese Remainder Theorem. Let

$$N = \mathfrak{o}_F \xi + \mathfrak{o}_F \eta \quad \text{and} \quad T_2 = \{v \in \Omega_F \setminus (T \cup \{v_1\} \cup T_1 \cup \infty_F) : N_v \text{ is not unimodular}\}$$

and

$$\delta_v = \begin{cases} \xi & v \in T \cup T_2 \\ \eta & v \in T_1. \end{cases}$$

Since

$$-\det(N)(F_{v_1}^\times)^2 = -\det(K)(F_{v_1}^\times)^2 \neq (F_{v_1}^\times)^2$$

for sufficiently small ϵ by our choice of v_1 and approximation, one can apply Step 1 to N with given vectors δ_v for $v \in T \cup T_1 \cup T_2$. For sufficiently small $\epsilon > 0$, there are $u \in N$ such that $|u - \delta_v|_v < \epsilon$ for all $v \in T \cup T_1 \cup T_2$ and $\text{ord}_v(q(u)) = 0$ for all primes $v \notin T \cup T_1 \cup T_2$ except one prime v_0 and $\text{ord}_{v_0}(q(u)) = 1$. By the approximation on T , one concludes that $u \in L_v + u_0$ for all $v \in \Omega_F$. Therefore $u \in L + u_0$. By approximation on $T_1 \cup T_2$, one also has that $\text{ord}_v(q(u)) = 0$ for all $v \in T_1 \cup T_2$. The proof is complete. \square

We extend §2 in [7] over a number field.

Lemma 4.4. *Suppose V_v is a non-degenerated quadratic space over F_v such that $\dim(V_v) \geq 3$ or $\dim(V_v) = 2$ with $-\det(V_v) \notin (F_v^\times)^2$ for $v < \infty_F$. Let L_v be a lattice in V_v and s be a positive integer. For any $x \in V_v$ with $x \neq 0$, there is an anisotropic vector $y \in V_v$ such that $x - y$ is anisotropic and*

$$x - y \in (F_v y)^\perp \cap \pi_v^s L_v.$$

Proof. If x is anisotropic, one can choose an anisotropic vector ξ such that

$$V_v = F_v x \perp F_v \xi \perp W$$

for some regular subspace W . Let t be a sufficiently large positive integer such that

$$\pi_v^t \xi \in \pi_v^s L_v, \quad q(\xi)q(x)^{-1}\pi_v^{2t}x \in \pi_v^s L_v \quad \text{and} \quad q(x) + \pi_v^{2t}q(\xi) \neq 0.$$

Let

$$y = (1 + q(\xi)q(x)^{-1}\pi_v^{2t})^{-1}(x + \pi_v^t \xi)$$

and y is anisotropic by the choice of t . Then

$$x - y = (1 + q(\xi)q(x)^{-1}\pi_v^{2t})^{-1}(q(\xi)q(x)^{-1}\pi_v^{2t}x - \pi_v^t \xi) \in \pi_v^s L_v$$

and $\langle x - y, y \rangle = 0$. Moreover,

$$q(x - y) = (1 + q(\xi)q(x)^{-1}\pi_v^{2t})^{-1} \cdot \pi_v^{2t} \cdot q(\xi) \neq 0$$

as required.

If x is isotropic, there is $w \in V_v$ such that $\langle x, w \rangle = 1$ and $q(w) = 0$. By the assumption on V_v , there are an anisotropic vector $h \in V_v$ and a regular subspace W of V_v such that

$$V_v = (F_v x + F_v w) \perp F_v h \perp W.$$

Let t be a sufficiently large positive integer such that

$$\pi_v^t h \in \pi_v^s L_v, \quad q(h)\pi_v^{2t}w \in \pi_v^s L_v \quad \text{and} \quad y = x - \pi_v^{2t}q(h)w + \pi_v^t h.$$

Then y is anisotropic and $\langle x - y, y \rangle = 0$ and $x - y \in \pi_v^s L_v$ as required. \square

Modifying the above result, one can extend the statement to $x = 0$.

Corollary 4.5. *Suppose V_v is a non-degenerated quadratic space over F_v such that $\dim(V_v) \geq 3$ or $\dim(V_v) = 2$ with $-\det(V_v) \notin (F_v^\times)^2$ for $v < \infty_F$. Let L_v be a lattice in V_v and s be a positive integer. If $x \in V_v$ with $x \neq 0$, there is an anisotropic vector $y \in V_v$ and a sublattice K_v in $(F_v y)^\perp \cap \pi_v^s L_v$ such that $x - y \in \pi_v^s L_v$ and $0 \neq q(x) - q(y)$ is represented by K_v primitively. Moreover, if V_v is isotropic, this is also true for $x = 0$.*

Proof. By lemma 4.4, one only needs to choose a sublattice K of $(F_v y)^\perp \cap \pi_v^s L_v$ with the maximal rank satisfying $x - y \in^* K$. In order to extend this statement to $x = 0$ for isotropic V_v , one can choose an isotropic vector $x' \in \pi_v^s L_v$ with $x' \neq 0$ and apply the previous result for x' . The result follows from $q(x') = q(0) = 0$. \square

Definition 4.6. *Suppose V_v is a non-degenerated quadratic space over F_v for $v < \infty_F$. Let L_v be a lattice in V_v and s be a positive integer. For any $x \in V_v$, we call the anisotropic vector y and the sublattice K_v in $(F_v y)^\perp \cap \pi_v^s L_v$ satisfying the property of Corollary 4.5 the associated vector and the associated lattice of x with respect to L_v and s respectively.*

It should be pointed out that Corollary 4.5 can not be extended to $x = 0$ if V_v is anisotropic since $-q(y)$ can not be represented by the space $(F_v y)^\perp$ by the anisotropic assumption.

Lemma 4.7. *Suppose V_v is a non-degenerated quadratic space over F_v for $v < \infty_F$. Let L_v be a lattice in V_v and $x \in V_v$. If y is sufficiently close to x , there is $\sigma \in O(L_v)$ such that $F_v \sigma x = F_v y$.*

Proof. Without loss of generality, one can assume $q(L_v) \subseteq \mathfrak{o}_v$ by scaling. Let $j \in \mathbb{Z}$ such that $\pi_v^j x$ is primitive in L_v^\sharp . If y is so close to x such that $\pi_v^j y$ is also primitive in L_v^\sharp and $q(y) = \epsilon^2 q(x)$ with $\epsilon \in \mathfrak{o}_v^\times$ and $\pi_v^j(\epsilon^{-1}y - x) \in L_v$, then there is $\tau \in O(V_v)$ such that $\tau(x) = \epsilon^{-1}y$ by Witt theorem. Applying Lemma 2.1 in [20], one obtains $\sigma \in O(L_v)$ such that $\mathfrak{o}_v \sigma x = \mathfrak{o}_v y$ as required. \square

One needs to modify the local cover in [7] to the following corollary.

Corollary 4.8. *Suppose V_v is a non-degenerated quadratic space over F_v such that $\dim(V_v) \geq 3$ or $\dim(V_v) = 2$ with $-\det(V_v) \notin (F_v^\times)^2$ for $v < \infty_F$. Let L_v be a lattice in V_v and s be a positive integer.*

If P is an open compact subset of V_v such that $0 \notin P$ if V_v is anisotropic, there are $\delta > 0$ and a finite subset $B \subset P$ and a finite set \mathfrak{B} of the associated vectors of B with respect to L_v and s satisfying the following property:

For any $x \in P$, there is $b \in B$ with the associated vector $\beta_b \in \mathfrak{B}$ such that $x - b \in \pi_v^s L_v$ and for any $|\xi - \beta_b|_v < \delta$ one has $\xi - x \in \pi_v^s L_v$ and there is a sublattice K_v of $\pi_v^s L_v \cap (F_v \xi)^\perp$ satisfying that whenever $|u - \beta_b|_v < \delta$ for $u \in V_v$ one has $q(x) - q(u)$ is represented by K_v primitively.

Proof. Let $\delta_0 > 0$ such that for any $|x - y|_v < \delta_0$ with $x, y \in V_v$ one has $x - y \in \pi_v^s L_v$.

For any $x \in P$, one fixes the associated vector β_x and the associated lattice $K_v(\beta_x)$ of x with respect to L_v and s by Corollary 4.5. Choose $0 < \delta(x, \beta_x) < \delta_0$ so small such that

1)

$$U(x, \delta(x, \beta_x)) = \{z \in V_v : |z - x|_v < \delta(x, \beta_x)\} \subseteq P$$

and $z - x \in \pi_v^s L_v$ for $z \in U(x, \delta(x, \beta_x))$.

2) if

$$|\xi - x|_v < \delta(x, \beta_x) \quad \text{and} \quad |\eta - \beta_x|_v < \delta(x, \beta_x)$$

for $\xi, \eta \in V_v$, one has $q(\xi) - q(\eta)$ is equal to $q(x) - q(\beta_x)$ up to a square of unit in \mathfrak{o}_v .

3) if $|z - \beta_x|_v < \delta(x, \beta_x)$, there is $\sigma \in O(L_v)$ such that $F_v \sigma \beta_x = F_v z$ by Lemma 4.7.

By compactness of P , there is a finite subset B of P such that $\{U(b, \delta(b, \beta_b))\}_{b \in B}$ is a cover of P . Define

$$\delta = \min_{b \in B} \{\delta(b, \beta_b)\}.$$

For any $x \in P$, there is $b \in B$ such that $x \in U(b, \delta(b, \beta_b))$. If $|\xi - \beta_b| < \delta \leq \delta_0$, one has

$$\xi - \beta_b \in \pi_v^s L_v \quad \text{and} \quad x - b \in \pi_v^s L_v$$

by the choice of δ_0 . By corollary 4.5, one has $b - \beta_b \in \pi_v^s L_v$. Therefore $\xi - x \in \pi_v^s L_v$.

By corollary 4.5, there is a sub-lattice $K(\beta_b)$ of maximal rank in $(F_v \beta_b)^\perp \cap \pi_v^s L_v$ such that $q(b) - q(\beta_b)$ is represented by $K(\beta_b)$ primitively. By 3), there is $\sigma \in O(L_v)$ such that $F_v \sigma \beta_b = F_v \xi$. Set

$$K_v = \sigma K(\beta_b) \subseteq \pi_v^s L_v \cap (F_v \xi)^\perp.$$

For any $|u - \beta_b| < \delta$ with $u \in V_v$, one has $q(x) - q(u)$ is equal to $q(b) - q(\beta_b)$ up to a square of unit in \mathfrak{o}_v by 2). Therefore $q(x) - q(u)$ is represented by $K_v = \sigma K(\beta_b)$ primitively. \square

When V_v is anisotropic, one can only apply Corollary 4.5 to $L_v \setminus \{0\}$ which is not compact. This fact produces the counter-example (9.1) in [1] where the local-global principle fails. In order to restrict to some compact subset of $L_v \setminus \{0\}$, one can consider, for example, the set of the primitive vectors of L_v .

Theorem 4.9. *Let V be a non-degenerated quadratic space over F and $w \in \infty_F$ such that V_w is positive definite and T be a finite set of non-archimedean primes containing all dyadic primes. If $\dim(V) = 4$, then Statement (LT) with respect to T holds.*

Proof. By Theorem 8.3 in [4], one only needs to consider that V_v is definite for all $v \in \infty_F$. We first prove the following claim.

Claim 4.10. For any non-archimedean prime $v_0 \notin T$, there is a positive number

$$C_1 = C_1(L, T, u_0, v_0)$$

such that whenever $\alpha \in F$ satisfies $\alpha > C_1$ at w and $\text{ord}_{v_0}(\alpha) = 0$ and $\alpha \rightarrow \text{gen}(L + u_0)$ with $\alpha \xrightarrow{*} L_v$ for those $v \in T$ where V_v is anisotropic and $u_0 \in L_v$, then there is $x \in L + u_0$ with $x \in {}^*L_v$ for all $v \notin T$ such that $q(x) = \alpha$.

Indeed, we put

$$P_v = \begin{cases} \{x \in {}^*L_v\} & V_v \text{ is anisotropic and } u_0 \in L_v \\ L_v + u_0 & \text{otherwise} \end{cases}$$

for all $v \in T$ and fix a positive integer s . Then there are $\delta_v > 0$ and a finite subset $B_v \subset P_v$ and a finite set \mathfrak{B}_v of the associated vectors of B_v with respect to L_v and s satisfying the property of Corollary 4.8. By the choice of P_v and Corollary 4.5, one has $\mathfrak{B}_v \subset P_v$ for $v \in T$. Let $\mathfrak{B}_{v_0} = \{x_{v_0}, y_{v_0}\} \subset L_{v_0}$ such that $q(x_{v_0}) = 1$ and $q(y_{v_0})$ is a non-square unit in \mathfrak{o}_{v_0} . Choose $\delta_{v_0} > 0$ so small such that if $|x - \lambda_{v_0}|_{v_0} < \delta_{v_0}$ for $\lambda_{v_0} \in \mathfrak{B}_{v_0}$ then

$$q(\lambda_{v_0})q(x)^{-1} \in (\mathfrak{o}_{v_0}^\times)^2.$$

Set

$$S = T \cup \{v_0\} \quad \text{and} \quad \Lambda = \prod_{v \in S} \mathfrak{B}_v.$$

For any $\lambda = (\lambda_v)_{v \in S} \in \Lambda$, there is $u_\lambda \in L + u_0$ and one prime $v_\lambda \notin S$ such that $|u_\lambda - \lambda_v|_v < \delta_v$ for all $v \in S$ and

$$\text{ord}_v(q(u_\lambda)) = \begin{cases} 0 & v \notin S \cup \{v_\lambda\} \\ 1 & v = v_\lambda \end{cases}$$

by Lemma 4.3. There is a sublattice K_v of $(F_v u_\lambda)^\perp$ satisfying the property of Corollary 4.8 for $v \in T$. Construct the sublattice K_λ of $L \cap (F u_\lambda)^\perp$ as follows

$$(K_\lambda)_v = \begin{cases} L_v \cap (F_v u_\lambda)^\perp & \text{if } v \notin T \\ K_v & \text{if } v \in T \end{cases}$$

and $(K_\lambda)_{v_0}$ is unimodular by $\text{ord}_{v_0}(q(u_\lambda)) = 0$.

Applying Proposition 2.11 ii) to K_λ for primitive representation outside $\{v_0\}$ with $\lambda \in \Lambda$, one obtains a positive number h_λ for $\lambda \in \Lambda$. Let $h = \max_{\lambda \in \Lambda} h_\lambda$ and

$$\gamma = \min_{v \in \infty_F} \min\{|q(x)|_v : x(\neq 0) \in L' + u'_0 \text{ with } \text{cls}(L' + u'_0) \in \text{gen}(L + u_0)\}.$$

For any

$$\lambda \in \Lambda, \quad a_{v_0} \in (\mathfrak{o}_{v_0}/(\pi_{v_0}^{h+3}))^\times \quad \text{and} \quad a_{v_\lambda} \in \mathfrak{o}_{v_\lambda}/(\pi_{v_\lambda}),$$

there is $\xi = \xi(\lambda, a_{v_0}, a_{v_\lambda}) \in \mathfrak{o}_F$ such that

$$\begin{cases} |\lambda_v - \xi u_\lambda|_v = |\lambda_v - u_\lambda|_v & \text{and } (\xi - 1)u_0 \in L_v & v \in T \\ \text{ord}_{v_0}(\xi - a_{v_0}) > h & \text{and } \text{ord}_{v_0}(\xi - a_{v_0}) \equiv 1 \pmod{2} & v = v_0 \\ \xi \equiv a_{v_\lambda} \pmod{\pi_{v_\lambda}} & & v = v_\lambda \\ |q(\xi u_\lambda)|_v < \frac{1}{2}\gamma & & v \in \infty_F \setminus \{w\} \end{cases} \quad (4.11)$$

by Chinese Remainder Theorem. Let

$$C_1 = C_1(L, T, u_0, v_0) = \max_\xi \{|q(\xi u_\lambda)|_w\}$$

where $\xi = \xi(\lambda, a_{v_0}, a_{v_\lambda})$ runs over all the above chosen values.

Since $\alpha \rightarrow \text{gen}(L + u_0)$ with $\alpha \xrightarrow{*} L_v$ for those $v \in T$ where V_v is anisotropic and $u_0 \in L_v$, there is $y_v \in P_v$ with $q(y_v) = \alpha$ for all $v \in T$. Then there is $b_v \in B_v$ and the associated vector λ_v of b_v with respect to L_v and s for $v \in T$ satisfying the property of Corollary 4.8. Let $\lambda_{v_0} \in \mathfrak{B}_{v_0}$ such that

$$\alpha \cdot q(\lambda_{v_0})^{-1} \in (\mathfrak{o}_{v_0}^\times)^2$$

by the assumption for α . Then $\lambda = (\lambda_v)_{v \in S} \in \Lambda$. For such $\lambda \in \Lambda$, one has $u_\lambda \in L + u_0$ and the prime $v_\lambda \notin S$ and K_λ as above with

$$\text{ord}_{v_\lambda}(q(u_\lambda)) = \text{ord}_{v_\lambda}(\det(K_\lambda)) = 1.$$

If $\text{ord}_{v_\lambda}(\alpha) = 1$, there are $x \in \mathfrak{o}_{\lambda_v}^\times$ and $\eta_{v_\lambda} \in \mathfrak{o}_{\lambda_v}$ satisfying the following equation

$$\frac{\alpha}{q(u_\lambda)} - \eta_{v_\lambda}^2 \equiv \frac{-\Delta_{\lambda_v} \cdot \det(K_\lambda)}{q(u_\lambda)} x^2 \pmod{\pi_{\lambda_v}} \quad (4.12)$$

where Δ_{λ_v} is a fixed non-square unit in $\mathfrak{o}_{\lambda_v}^\times$. Choose

$$a_{v_\lambda} \equiv \begin{cases} \eta_{v_\lambda} & \text{ord}_{v_\lambda}(\alpha) = 1 \\ 1 & \text{otherwise} \end{cases} \pmod{\pi_{v_\lambda}}.$$

By the choice of δ_{v_0} , there is $\eta_{v_0} \in \mathfrak{o}_{v_0}^\times$ such that $\alpha = \eta_{v_0}^2 q(u_\lambda)$. Choose

$$a_{v_0} \equiv \eta_{v_0} \pmod{\pi_{v_0}^{h+3}}.$$

For the above chosen λ , a_{v_0} and a_{v_λ} , we get $\xi = \xi(\lambda, a_{v_0}, a_{v_\lambda})$ as (4.11).

Next we will show that $\alpha - q(\xi u_\lambda) \xrightarrow{*} (K_\lambda)_v$ for all $v \in \Omega_F$.

When $v \notin T \cup \{v_\lambda\} \cup \{\infty_F\}$, the claim follows from Lemma 4.2 in [5].

When $v = v_\lambda$, one has that the first Jordan component of $(K_\lambda)_{v_\lambda}$ is a unimodular lattice of rank $(\dim(V) - 2)$. One only needs to consider the first Jordan component of $(K_\lambda)_{v_\lambda}$ is of rank

2 and is not a hyperbolic plane. Then the discriminant of the first Jordan component is $-\Delta_{\lambda_v}$. If α is a unit, then

$$\alpha - q(\xi u_\lambda) \xrightarrow{*} (K_\lambda)_{v_\lambda}$$

by Hensel's Lemma. If $\text{ord}_{v_\lambda}(\alpha) \geq 1$, then $\text{ord}_{v_\lambda}(\alpha - q(\xi u_\lambda)) = 1$ and $\alpha - q(\xi u_\lambda)$ is represented by $(K_\lambda)_{v_\lambda}$ by the equation (4.12). This implies that

$$\alpha - q(\xi u_\lambda) \xrightarrow{*} (K_\lambda)_{v_\lambda}$$

by the order consideration of the coefficients.

When $v \in T$, then

$$\alpha - q(\xi u_\lambda) \xrightarrow{*} (K_\lambda)_v$$

by Corollary 4.8.

When $v \in \infty_F \setminus \{w\}$, then $\alpha - q(\xi u_\lambda)$ and α have the same sign at v by the choice of γ . Therefore $\alpha - q(\xi u_\lambda)$ is represented by $F_v K_\lambda$.

When $v = w$, the assumption that $\alpha > C_1$ at w implies that $\alpha - q(\xi u_\lambda)$ is positive at w . Therefore $\alpha - q(\xi u_\lambda)$ is represented by $F_w K_\lambda$.

Since $\text{ord}_{v_0}(\alpha - q(\xi u_\lambda)) > h$ and $\text{ord}_{v_0}(\alpha - q(\xi u_\lambda)) \equiv 1 \pmod{2}$ by (4.11), one obtains $z \in K_\lambda$ with $z \in^* (K_\lambda)_v$ for $v \neq v_0$ such that $q(z) = \alpha - q(\xi u_\lambda)$ by applying Corollary 2.9 and Proposition 2.11 to K_λ and $\{v_0\}$. Therefore $\alpha = q(\xi u_\lambda + z)$ with $\xi u_\lambda + z \in L + u_0$. Since

$$(K_\lambda)_v = L_v \cap (F_v u_\lambda)^\perp$$

for $v \notin T$, one concludes that $\xi u_\lambda + z \in^* L_v$ for all $v \notin T$ and $v \neq v_0$. One also has $\xi u_\lambda + z \in^* L_{v_0}$ by $\text{ord}(\alpha) = \text{ord}(q(\xi u_\lambda + z)) = 0$. We complete the proof of Claim 4.10.

Finally we need to modify Cassels' trick (see Lemma 9.1 of Chapter 11 in [1]) over a number field to complete the proof by using Claim 4.10. Choose a non-archimedean $v_0 \notin T$ such that $d(L_{v_0}) \in (\mathfrak{o}_{v_0}^\times)^2$ and v_0 is a principle ideal of \mathfrak{o}_F by the Chebatorev density theorem. Let V^α be the quadratic space obtained by scaling V with $\alpha \in F^\times$. Fix a generator π_{v_0} of prime ideal v_0 of \mathfrak{o}_F such that $0 < \pi_{v_0} < 1$ at w . Then

$$V \cong V^{\pi_{v_0}^{2m}} \quad \text{and} \quad V^{\pi_{v_0}} \cong V^{\pi_{v_0}^{2m+1}}$$

over F for all $m \in \mathbb{Z}$ by 66:5 Remark in [16]. Since there are only finitely many classes with the given scale and volume in a non-degenerated quadratic space by 103:4 Theorem in [16], one concludes that the set

$$\mathfrak{C} = \{cls(K + u_0) : K \text{ is a lattice in } V^{\pi_{v_0}^m} \text{ with } \mathfrak{s}(K) = \mathfrak{s}(L) \text{ and } \mathfrak{v}(K) = \mathfrak{v}(L), \forall m \in \mathbb{Z}\}$$

is finite, where $\mathfrak{s}(K)$ and $\mathfrak{s}(L)$ are the scales of K and L respectively and $\mathfrak{v}(K)$ and $\mathfrak{v}(L)$ are the volumes of K and L respectively defined as §82E in [16]. Let

$$c = c(L, T, u_0) = \max_{cls(K+u_0) \in \mathfrak{C}} \{C_1(L, T, u_0, v_0) \text{ as Claim 4.10}\}.$$

If $\alpha \rightarrow \text{gen}(L + u_0)$ with $\alpha > c$ at w and $\alpha \xrightarrow{*} L_v$ when V_v is anisotropic and $u_0 \in L_v$ for $v \in T$, there is a positive integer k such that $\text{ord}_{v_0}(\pi_{v_0}^{-k} \alpha) = 0$ and $\pi_{v_0}^{-k} \alpha > \alpha > c$ at w . Since

$$L_{v_0} = (\mathfrak{o}_{v_0} e_1 + \mathfrak{o}_{v_0} f_1) \perp (\mathfrak{o}_{v_0} e_2 + \mathfrak{o}_{v_0} f_2)$$

with $q(e_1) = q(f_1) = q(e_2) = q(f_2) = 0$ and $\langle e_1, f_1 \rangle = \langle e_2, f_2 \rangle = 1$, one defines $M \subset L$ as follows

$$M_v = \begin{cases} (\mathfrak{o}_{v_0}e_1 + \mathfrak{o}_{v_0}\pi_{v_0}^k f_1) \perp (\mathfrak{o}_{v_0}e_2 + \mathfrak{o}_{v_0}\pi_{v_0}^k f_2) & v = v_0 \\ L_v & v \neq v_0. \end{cases}$$

Then M is a lattice in $V^{\pi_{v_0}^{-k}}$ such that $cls(M + u_0) \in \mathfrak{C}$. Applying Claim 4.10 for $\pi_{v_0}^{-k}\alpha$ and $M + u_0$ in $V^{\pi_{v_0}^{-k}}$, one gets $x \in (M + u_0) \subset (L + u_0)$ with $x \in^* M_v$ for $v \notin T$ such that $q(x) = \alpha$. Since $ord_{v_0}(\alpha) = k$, one concludes that $x \in^* L_{v_0}$ as well. The proof is complete. \square

The weak version of Theorem 4.9 was also proved in Theorem 4.9 of [2].

Remark 4.13. *It should be pointed out that Cassels' trick (Lemma 9.1 of Chapter 11 in [1]) can be applied only for even dimensional quadratic spaces. For Statement (LT) with $\dim(V) \geq 5$, one needs a slight different proof (see p.236-p.241 in [1] for weak version). In [11], Jöchner and Kitaoka generalized such result to a lattice of rank n represented by a lattice of rank $m \geq 2n + 3$ with congruent condition under a bounded restriction. In [10], Jöchner further removed such bounded restriction for $n = 2$.*

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